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Generalization of Sato equation and systems of multidimensional nonlinear Partial Differential Equations

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Abstract

This paper develops one of the methods for study of nonlinear Partial Differential equations. We generalize Sato equation and represent the algorithm for construction of some classes of nonlinear Partial Differential Equations (PDE) together with solutions parameterized by the set of arbitrary functions.

1 Introduction

Nonlinear Partial Differential Equations (PDE) have a great variety of application in both mathematical physics and applied mathematics. The well developed tools for study are admitted by the wide class of PDE which is called Completely Integrable Nonlinear PDE. These systems reflect many important properties of real physical objects and have been studied in many details. So, Korteweg-de Vries equation (KdV) [1, 2, 3, 4], Kadomtsev-Petviashvili equation (KP) [4], Camassa-Holm equation [5, 6, 7, 8] and some of its two

dimensional generalizations (for instance, [9]), are well known in hydrodynamics, where they describe propagation of surface waves on incompressible fluid without viscosity; Nonlinear Schrödinger equation (NLS) is closely used in nonlinear fiber optics, where it describes propagation of electromagnetic pulses [10, 11]. Note that some methods for study of the integrable PDE do not presume in advance their integrability. For instance, Sato theory [12, 13, 14], Hirota bilinear method [15, 16, 17, 18], Penlevé method [19, 20, 21], recent modification of the dressing method based on the algebraic matrix equation [22, 23, 24].

In this paper we use the ideas of the ref.[12], where the space of one dimensional pseudo-differential (or differential) operators has been used for description of the KP hierarchy. We replace the space of one dimensional operators with the space of multi-dimensional *differential* operators, which allows one to treat larger class of PDE, in particular, $(n+1)$-dimensional PDE with $n > 2$. We provide the family of particular solutions, parameterized by the set of arbitrary functions. Although we have not answered the question about complete integrability of the PDE under consideration. In comparison with approach considered in the refs. [22, 23, 24], the manifold of available solutions is reacher. Trying to make the understanding of this article independent on references, we supply all necessary details below.

General idea is developed in Sec.2. The useful subspace of n-order multi-dimensional differential operators together with related systems of nonlinear PDE is considered in the Sec.3. We represent simple example of (2+1)-dimensional equation and generalization of the KP hierarchy in the same section. Algorithm for construction of the families of particular solutions is discussed in the Sec.4. Conclusions are given in the Sec.5.

2 Space of N-dimensional differential operators and related systems of nonlinear PDE

Hereafter we will use Greek letters to denote multi-index, $\alpha = (\alpha_1 \dots \alpha_N)$, where N is the dimension of x-space. The dimension of t-space (space of additional parameters t_γ, $\gamma = (\gamma_1, \dots \gamma_N)$) may be arbitrary. In particular,

it may be infinity. By definition,

$$x = (x_1, \ldots, x_N), \quad ||\alpha|| = \sum_{i=1}^{N} \alpha_i, \tag{1}$$

$$\partial^\alpha = \prod_{i=1}^{N} \partial_i^{\alpha_i}, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad \partial_\gamma \equiv \frac{\partial}{\partial t_\gamma},$$

$$Q_n^\alpha \;\; \text{is the number of all possible values of} \;\; \alpha \;\; \text{with} \;\; ||\alpha|| = n. \tag{2}$$

All functions are differentiable with respect to their arguments as many times as one needs.

Denote $\mathbb{P}^n$ the linear space of n-order differential operators of the following general form

$$W_n \in \mathbb{P}^n : W_n = \sum_{||\beta||=0}^{n} w_\beta \partial^\beta, \tag{3}$$

where w_β are functions of x and t, which will be specified below. Denote Q_n the number of terms in the operator W_n of general form (3):

$$Q_n = \sum_{||\alpha||=0}^{n} Q_n^\alpha. \tag{4}$$

Space $\mathbb{P}^n$ is related with Q_n-dimensional linear space $\mathbb{C}^{Q_n}$ of functions $\psi(x,t)$ with the basis ψ_i, $i = 1, \ldots, Q_n$, satisfying the following condition:

$$\det||\partial^\beta \psi_i|| \neq 0, \quad 0 \leq ||\beta|| \leq n, \quad i = 1, \ldots, Q_n, \tag{5}$$

where $||\partial^\beta \psi_i||$ is $Q_n \times Q_n$ matrix. Indexes α and i enumerate columns and rows of this matrix respectively. By definition, dependence on parameters x is arbitrary, while parameters t_γ are introduced in functions ψ_i by the following formulas:

$$\partial_\gamma \psi_i = \partial^\gamma \psi_i. \tag{6}$$

Suggestion 1. Let parameter Q_n be defined by the eq.(4). For any α such that $||\alpha|| = n$ and any space $\mathbb{C}^{Q_{n-1}}$ there is set of Q_{n-1} *uniquely* defined

functions w^α_β such that space $\mathbb{C}^{Q_{n-1}}$ forms Q_{n-1}-dimensional linear subspace of solutions for the following linear differential equation:

$$S_\alpha \psi = 0, \quad S_\alpha = \partial^\alpha + W_\alpha, \quad W_\alpha = \sum_{||\beta||=0}^{n-1} w^\alpha_\beta \partial^\beta \in \mathbb{P}^n. \tag{7}$$

$\triangle$ Let us fix parameter α with $||\alpha|| = n$ and basis ψ_i, $i = 1, \ldots, Q_{n-1}$ of space $\mathbb{C}^{Q_{n-1}}$. One can write the formal system of differential equations (7)

$$S_\alpha \psi_i \equiv \partial^\alpha \psi_i + \sum_{||\beta||=0}^{n-1} w^\alpha_\beta \psi_i = 0, \quad i = 1, \ldots, Q_{n-1}. \tag{8}$$

Let us consider the system (8) as nonhomogeneous system of *algebraic* equations on the functions w^α_β. The matrix of this system, $||\partial^\beta \psi_i||$ $(0 \leq ||\beta|| \leq n-1,\ i = 1, \ldots, Q_{n-1})$, is nondegenerate due to the condition (5). So that this system has *unique* solution, i.e. functions w^α_β are defined uniquely in terms of functions ψ_i and their derivatives. Owing to this, w^α_β are functions of both sets of variables, x and t. By construction, we have determined coefficients w^α_β in the system of differential equations (7), for which given set of functions ψ_i $(i = 1, \ldots, Q_{n-1})$ forms basis for certain solution subspace. $\blacktriangle$

Equations of the form (7) with all possible values of parameter α such that $||\alpha|| = n$, form system of Q^α_n differential equations. One can show that its coefficients w^α_β satisfy some system of nonlinear PDE. For this purpose, one needs to fix multi-index γ, $||\gamma|| = n_\gamma$, differentiate the eqs.(7) with respect to t_γ and study the result:

$$\left(\frac{\partial S_\alpha}{\partial t_\gamma} + S_\alpha \partial^\gamma \right) \psi = 0, \quad ||\gamma|| = n_\gamma. \tag{9}$$

Suggestion 2. For each particular pair of parameters α $(||\alpha|| = n)$ and γ there is set of Q^α_n differential operators $B_{\gamma\alpha\epsilon} = \sum_{||\delta||=0}^{n_\gamma} v_{\gamma\alpha\epsilon\delta} \partial^\delta \in \mathbb{P}^{n_\gamma}$ with coefficients expressed in terms of functions w^α_β and their derivatives such that operators $M_{\gamma\alpha}$

$$M_{\gamma\alpha} \equiv \frac{\partial S_\alpha}{\partial t_\gamma} + S_\alpha \partial^\gamma + \sum_{||\epsilon||=n} B_{\gamma\alpha\epsilon} S_\epsilon \tag{10}$$

belong to the space $\mathbb{P}^{n-1}$, i.e. $M_{\gamma\alpha}$ can be written in the form

$$M_{\gamma\alpha} = \sum_{||\beta||=0}^{n-1} a_\beta^{\gamma\alpha}\partial^\beta, \tag{11}$$

where $a_\beta^{\gamma\alpha}$ are some functions of ω_β^α and their derivatives.

$\triangle$ The first term in the operator $M_{\gamma\alpha}$, $\dfrac{\partial S_\alpha}{\partial t_\gamma} \equiv \displaystyle\sum_{||\beta||=0}^{n-1} \frac{\partial w_\beta^\alpha}{\partial t_\gamma}\partial^\beta$, belongs to the space $\mathbb{P}^{n-1}$. Since $S^\alpha \in \mathbb{P}^n$, the second term in the eq.(10) belongs to the space $\mathbb{P}^{n+n_\gamma}$. Let $B_{\gamma\alpha\epsilon} = \sum_{||\delta||=0}^{n_\gamma} v_{\gamma\alpha\epsilon\delta}\partial^\delta \in \mathbb{P}^{n_\gamma}$ be n_γ-order differential operators with arbitrary coefficients. Then one can write

$$\begin{aligned}
\sum_{||\epsilon||=n} B_{\gamma\alpha\epsilon}S_\epsilon &= \sum_{||\beta||=n}^{n+n_\gamma} b_{\gamma\alpha\beta}\partial^\beta + R^{n-1}, \quad R^{n-1} \in \mathbb{P}^{n-1}, && (12)\\
b_{\gamma\alpha\beta} &= v_{\gamma\alpha\epsilon(\beta-\epsilon)} + && (13)\\
&\quad f(v_{\gamma\alpha\epsilon\tilde\delta},\ ||\tilde\delta|| > ||\beta-\epsilon||; \text{coefficients of operators } W_\alpha),\\
&\quad ||\beta|| = n,\dots,n+n_\gamma.
\end{aligned}$$

This means that $\sum_{||\epsilon||=n} B_{\gamma\alpha\epsilon}S_\epsilon \in \mathbb{P}^{n+n_\gamma}$ and has all arbitrary coefficients ahead of the derivatives of the order grater or equal to n. Let us choose these coefficients in such a way that

$$\left(S_\alpha\partial^\gamma + \sum_{||\beta||=n} B_{\gamma\alpha\epsilon}S_\epsilon\right) \in \mathbb{P}^{n-1}. \tag{14}$$

which provides condition $M_{\gamma\alpha} \in \mathbb{P}^{n-1}$. Due to the relation (14), coefficients $v_{\gamma\alpha\epsilon\delta}$ are expressed in terms of the coefficients of the operators W_α, $||\alpha|| = n$. $\blacktriangle$

By construction, operators $M_{\gamma\alpha}$ generate the following system of $Q_{n-1} \times Q_n^\alpha$ linear differential equations:

$$M_{\gamma\alpha}\psi_i \equiv \sum_{||\beta||=0}^{n-1} a_\beta^{\gamma\alpha}\partial^\beta\psi_i = 0, \quad ||\alpha|| = n, \quad i = 1,\dots,Q_{n-1}. \tag{15}$$

Let us regard the system (15) as linear *homogeneous* system of algebraic equations on coefficients $a^{\gamma\alpha}_{\beta}$ with nondegenerate matrix $||\partial^{\beta}\psi_i||$ ($0 \le ||\beta|| \le n-1$) due to the condition (5). This system has only trivial solution, i.e.

$$a^{\gamma\alpha}_{\beta} = 0. \tag{16}$$

From another point of view, coefficients $a^{\gamma\alpha}_{\beta}$ have been defined in terms of coefficients of the operators W_α and their derivatives, i.e. the system (16) is system of nonlinear PDE on functions w^{α}_{β}, generated by the space of $(n-1)$-order differential operators W_α with parameter t_γ introduced by the eq.(6). The complete set of systems of nonlinear PDE, related with all possible parameters t_γ, introduced by the eq.(6), forms hierarchy of PDE associated with given space $\mathbb{P}^{n-1}$.

We call the equation (15) written in the form

$$\frac{\partial S_\alpha}{\partial t_\gamma} + S_\alpha \partial^\gamma + \sum_{||\epsilon||=n} B_{\gamma\alpha\epsilon} S_\epsilon = 0, \quad ||\alpha|| = n \tag{17}$$

generalization of Sato equation. The system of equations (8) serves for construction of particular solutions to the system of nonlinear PDE (16), which will be used in Sec.4.

The system (16) consists of $Q_{n-1} \times Q^{\alpha}_{n}$ equations. Equations of the system (16) with any particular γ recursively relate all $Q_{n-1} \times Q^{\alpha}_{n}$ coefficients w^{α}_{β} ($||\alpha|| = n$, $||\beta|| = 0, 1, \dots, n-1$) of the operators W_α and compose the complete system of equations for whole set of these coefficients. Disadvantage of this system is that its structure depends on n. In the next section we consider the subspace of the space $\mathbb{P}^{n-1}$, which allows one to generate the systems of PDE which do not depend on the value of parameter n, if only this value is big enough. These systems of PDE admit the class of solutions, which depends on the set of arbitrary functions of single variable.

3 About reductions

Let $\mathcal{P}^{n}_{n_2,\dots,n_N} \subset \mathbb{P}^n$, $n = \sum_{i=1}^{N} n_i$, be defined as follows:

$$W_n \in \mathcal{P}^{n}_{n_2,\dots,n_N} : W_n = \sum_{||\beta||=0}^{n} w_{\hat\beta} \partial^\beta, \quad \beta_i \le n_i \;\; \text{for} \;\; i > 1, \tag{18}$$

$$\beta = (\beta_1 \beta_2 \dots \beta_N), \quad \hat\beta = ((n - \beta_1 - 1)\beta_2 \dots \beta_N),$$

i.e. parameters n_i, $i = 1, \ldots, N$, mean the order of the highest derivative with respect to x_i in the differential operators W_n. Emphasize that we don't put any restriction on the order of the derivative with respect to x_1, i.e its highest possible order in operators W_n equals n. Index with hat is introduced for convenience of representation of system of nonlinear PDE. All multi-indexes (without hat) in this section have the following structure:

$$\alpha = (\alpha_1 \ldots \alpha_N) \quad \text{with} \;\; \alpha_i < n_i \quad \text{for} \;\; i > 1. \tag{19}$$

Let us introduce the set of equations, which specifies the dependence on x_i, $i > 1$:

$$\partial_i^{n_i+1} \psi = \partial^{\alpha^{(i)}} \psi, \;\; \alpha^{(i)} = (\alpha_1^{(i)} \ldots \alpha_N^{(i)}), \;\; \alpha_i^{(i)} = 0, \tag{20}$$

where $\psi \in \mathbb{C}^{Q_n}$. Recall that Q_n is the maximum number of terms in operators $W_n \in \mathcal{P}^n_{n_2,\ldots,n_N}$. Due to the relations (20), one can use variables x_i ($i > 1$) along with t_γ for construction of the nonlinear eqs. (16). One needs just replace ∂_γ with ∂_i in formulas (9-17). This fact will be used in sections 3.1.1 and 3.1.2.

3.1 Examples of nonlinear PDE related with space $\mathcal{P}^n_1$

In this section we will use two-dimensional x-space, $x = (x_1, x_2)$, $W_j \in \mathcal{P}^{n-1}_1$, $j = 1, 2, \ldots$. Maximum order of derivative with respect to x_2 in the operators W_j equals one: $n_2 = 1$. Because of that, the set of operators S_α consists of two operators $S_1 \equiv S_{n0}$ and $S_2 \equiv S_{(n-1)1}$ for any fixed parameter n:

$$S_1 \equiv \partial_1^n + W_1, \;\; W_1 = \sum_{k=0}^{n-2} u_{(n-k-2)1} \partial_1^k \partial_2 + \sum_{k=0}^{n-1} u_{(n-k-1)0} \partial_1^k, \tag{21}$$

$$S_2 \equiv \partial_1^{n-1} \partial_2 + W_2, \;\; W_2 = \sum_{k=0}^{n-2} v_{(n-k-2)1} \partial_1^k \partial_2 + \sum_{k=0}^{n-1} v_{(n-k-1)0} \partial_1^k. \tag{22}$$

These operators introduce two differential equations (7) on the function ψ:

$$S_1 \psi = 0, \quad S_2 \psi = 0. \tag{23}$$

It is simple to define the number of terms Q_{n-1} in the operators W_j:

$$Q_{n-1} = 2n - 1. \tag{24}$$

Next, one needs to fix $(2n-1)$-dimensional basis ψ_i $(i=1,\dots,2n-1)$ in subspace $\mathbb{C}^{2n-1}$ and write formally the system (8)

$$\begin{aligned} S_1\psi_i &= 0, && (25)\\ S_2\psi_i &= 0, \quad i=1,\dots,2n-1, && (26)\end{aligned}$$

Condition (5) now reads

$$\left|\begin{array}{ccccccc} \partial_1^{n-1}\psi_1 & \partial_1^{n-2}\psi_1 & \cdots & \psi_1 & \partial_1^{n-2}\partial_2\psi_1 & \cdots & \partial_2\psi_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial_1^{n-1}\psi_{2n-1} & \partial_1^{n-2}\psi_{2n-1} & \cdots & \psi_{2n-1} & \partial_1^{n-2}\partial_2\psi_{2n-1} & \cdots & \partial_2\psi_{2n-1} \end{array}\right| \neq 0 \quad (27)$$

Each of the systems (25) and (26) should be treated as system of algebraic equations on coefficients u_{ij} and v_{ij} respectively. Due to the condition (27) these equations have unique nontrivial solution.

3.1.1 Example 1: (2+1) dimensional PDE

In this example we take the eq.(20) in the following form

$$\partial_2^2\psi = \partial_1\psi, \quad (28)$$

and use notations

$$x \equiv x_1, \quad t_1 \equiv x_2, \quad t_2 \equiv t_\alpha, \quad \alpha = (20). \quad (29)$$

Function ψ depends on variable t_2 due to the formula (see eq. (6))

$$\partial_{t_2}\psi = \partial_1^2\psi. \quad (30)$$

Equations (17) related with parameters $t_1 \equiv x_2$ and t_2 have the following form:

$$M_{1j} = \frac{\partial S_j}{\partial x_2} + S_j\partial_1 + B_{1j1}S_1 + B_{1j2}S_2 = 0, \quad j=1,2, \quad (31)$$

$$M_{2j} = \frac{\partial S_j}{\partial t_2} + S_j\partial_1^2 + B_{2j1}S_1 + B_{2j2}S_2 = 0, \quad j=1,2, \quad (32)$$

where B_{ijk} are differential operators of the next form

$$\begin{aligned} B_{111} &= v_{00}, \quad B_{112} = (-u_{00}+v_{01}) - \partial_1, && (33)\\ B_{121} &= -1, \quad B_{122} = -v_{00}, && (34)\\ B_{211} &= 2u_{00x} - \partial_1^2, \quad B_{212} = 2u_{01x}, && (35)\\ B_{221} &= 2v_{00x}, \quad B_{222} = 2v_{01x} - \partial_1^2, && (36)\end{aligned}$$

which provide the following structure of the operators M_{ij} (see (11)):

$$M_{ij}=\sum_{s=0}^{n-2}a^{ij}_{s1}\partial_1^s\partial_2+\sum_{s=0}^{n-1}a^{ij}_{s0}\partial_1^s,\quad i,j=1,2 \tag{37}$$

and

$$a^{ij}_{sk}\equiv 0 \tag{38}$$

(compare with derivation of the eq.(16)).

Below are several equations from the list (38) which compose the complete system of equations:

$$\begin{aligned}
a^{11}_{(n-1)0} &= u_{01}+v_{00}v_{01}-v_{10}+u_{00x_2}-v_{00x}=0, && (39)\\
a^{11}_{(n-2)1} &= u_{10}+u_{01}v_{00}-u_{00}v_{01}+v_{01}^2-v_{11}+u_{01x_2}-v_{01x}=0, && (40)\\
a^{12}_{(n-1)0} &= -u_{00}-v_{00}^2+v_{01}+v_{00x_2}=0, && (41)\\
a^{12}_{(n-2)1} &= -u_{01}-v_{00}v_{01}+v_{10}+v_{01x_2}=0, && (42)\\
a^{12}_{(n-2)0} &= -u_{10}-v_{00}v_{10}+v_{11}+v_{10x_2}=0, && (43)\\
a^{21}_{(n-1)0} &= u_{00t_2}+2u_{00}u_{00x}+2v_{00}u_{01x}-2u_{10x}-u_{00xx}=0, && (44)\\
a^{21}_{(n-2)1} &= u_{01t_2}+2u_{01}u_{00x}+2v_{01}u_{01x}-2u_{11x}-u_{01xx}=0, && (45)\\
a^{22}_{(n-1)0} &= v_{00t_2}+2u_{00}v_{00x}+2v_{00}v_{01x}-2v_{10x}-v_{00xx}=0, && (46)\\
a^{22}_{(n-2)1} &= v_{01t_2}+2u_{01}v_{00x}+2v_{01}v_{01x}-2v_{11x}-v_{01xx}=0, && (47)
\end{aligned}$$

We leave without proof the statement that this system remains the same for any value of parameter n, which happens due to the special structure of the subspace $\mathcal{P}_1^{n-1}$.

To simplify the system (39-47), let us solve eqs. (41,42,43) with respect to u_{00}, u_{01} and u_{10} respectively, substitute the result in the rest of equations. Then eqs.(46) and (39,40) result in the following system of equations on the functions $u=v_{00}$, $v=v_{01}$, and $w=v_{10}$:

$$\begin{aligned}
u_{t_2}-2u^2u_x+2vu_x+2u_{x_2}u_x+2uv_x-2w_x-u_{xx} &= 0,\\
-2uu_{x_2}+2v_{x_2}+u_{x_2x_2}-u_x &= 0,\\
-2vu_{x_2}+2w_{x_2}+v_{x_2x_2}-v_x &= 0.
\end{aligned} \tag{48}$$

Simple example of particular solution for this system will be constructed in the Sec.4.

3.1.2 Example 2: (3+1)-dimensional generalization of KP hierarchy

In this example we use the following notations:

$$x \equiv x_1, \;\; t_1 \equiv x_2, \;\; t_i \equiv t_{\alpha^{(i)}}, \;\; i = 2,3, \;\; \alpha^{(2)} = (11), \;\; \alpha^{(3)} = (30). \tag{49}$$

Dependence on $t_1 = x_2$, t_2 and t_3 is given by the following equations:

$$\partial_2^2 \psi = \partial_1^2 \psi, \;\; \partial_{t_2} \psi = \partial_1 \partial_2 \psi, \;\; \partial_{t_3} \psi = \partial_1^3 \psi, \tag{50}$$

Equations (17) have the following form:

$$M_{1j} = \frac{\partial S_j}{\partial x_2} + S_j \partial_1^2 + B_{1j1} S_1 + B_{1j2} S_2 = 0, \;\; j = 1,2, \tag{51}$$

$$M_{2j} = \frac{\partial S_j}{\partial t_2} + S_j \partial_1 + B_{2j1} S_1 + B_{2j2} S_2 = 0, \;\; j = 1,2, \tag{52}$$

$$M_{3j} = \frac{\partial S_j}{\partial t_3} + S_j \partial_1 + B_{3j1} S_1 + B_{3j2} S_2 = 0, \;\; j = 1,2, \tag{53}$$

One has the following expressions for differential operators B_{ijk}:

$$B_{111} = (-u_{01} + v_{00}), \tag{54}$$

$$B_{112} = (-u_{00} + v_{01}) - \partial_1, \tag{55}$$

$$B_{121} = (u_{00} - v_{01}) - \partial_1, \tag{56}$$

$$B_{122} = (u_{01} - v_{00}), \tag{57}$$

$$B_{211} = (u_{00}u_{01} - u_{11} - v_{00}v_{01} + v_{10} + 2v_{00x}) + (-u_{01} + v_{00})\partial_1, \tag{58}$$

$$B_{212} = (u_{01}^2 - u_{10} - u_{01}v_{00} + u_{00}v_{01} - v_{01}^2 + v_{11} + 2v_{01x}) + (-u_{00} + v_{01})\partial_1 - \partial_1^2, \tag{59}$$

$$B_{221} = (-u_{00}^2 + u_{10} - u_{01}v_{00} + v_{00}^2 + u_{00}v_{01} - v_{11} + 2u_{00x}) + (u_{00} - v_{01})\partial_1 - \partial_1^2, \tag{60}$$

$$B_{222} = (-(u_{00}u_{01}) + u_{11} + v_{00}v_{01} - v_{10} + 2u_{01x}) + (u_{01} - v_{00})\partial_1, \tag{61}$$

$$B_{311} = -3(u_{00}u_{00x} + v_{00}u_{01x} - u_{10x} - u_{00xx}) + 3u_{00x}\partial_1 - \partial_1^3, \tag{62}$$

$$B_{312} = -3(u_{01}u_{00x} + v_{01}u_{01x} - u_{11x} - u_{01xx}) + 3u_{01x}\partial_1, \tag{63}$$

$$B_{321} = -3(u_{00}v_{00x} + v_{00}v_{01x} - v_{10x} - v_{00xx}) + 3v_{00x}\partial_1, \tag{64}$$

$$B_{322} = -3(u_{01}v_{00x} + v_{01}v_{01x} - v_{11x} - v_{01xx}) + 3v_{01x}\partial_1 - \partial_1^3. \tag{65}$$

With given operators B_{ijk}, equations (51-53) take the form (37) with $i = 1,2,3$ and $j = 1,2$. Write down several nonlinear equations, generated by each of the eqs.(51-53).

From the eq.(51):

$$\begin{aligned}
a^{11}_{(n-1)0} &= -(u_{00}u_{01}) + u_{11} + v_{00}v_{01} - v_{10} + u_{00x_2} - v_{00x} = 0, && (66)\\
a^{11}_{(n-2)1} &= -u_{01}^2 + u_{10} + u_{01}v_{00} - u_{00}v_{01} + v_{01}^2 - v_{11} + && (67)\\
& \quad u_{01x_2} - v_{01x} = 0,\\
& \quad \ldots\ldots\ldots\ldots
\end{aligned}$$

From the eq.(52):

$$\begin{aligned}
a^{21}_{(n-1)0} &= u_{00}^2u_{01} - u_{01}u_{10} - u_{00}u_{11} + u_{21} + u_{01}^2v_{00} - u_{01}v_{00}^2 - && (68)\\
& \quad v_{00}v_{01}^2 + v_{01}v_{10} + v_{00}v_{11} - v_{20} + u_{00t_2} - u_{01}u_{00x} +\\
& \quad v_{00}u_{00x} + u_{00}v_{00x} + v_{01}v_{00x} + 2v_{00}v_{01x} -\\
& \quad 2v_{10x} - v_{00xx} = 0,\\
a^{21}_{(n-2)1} &= u_{00}u_{01}^2 - 2u_{01}u_{11} + u_{20} + u_{11}v_{00} + u_{01}^2v_{01} - && (69)\\
& \quad u_{10}v_{01} - 2u_{01}v_{00}v_{01} + u_{00}v_{01}^2 - v_{01}^3 + u_{01}v_{10} -\\
& \quad u_{00}v_{11} + 2v_{01}v_{11} - v_{21} + u_{01t_2} - u_{01}u_{01x} + v_{00}u_{01x} +\\
& \quad 2u_{01}v_{00x} - u_{00}v_{01x} + 3v_{01}v_{01x} - 2v_{11x} - v_{01xx} = 0,\\
& \quad \ldots\ldots\ldots\ldots
\end{aligned}$$

From the eq.(53):

$$\begin{aligned}
a^{31}_{(n-1)0} &= u_{00t_3} - 3u_{00}^2u_{00x} + 3u_{10}u_{00x} - 3u_{01}v_{00}u_{00x} + 3u_{00x}^{\ 2} - && (70)\\
& \quad 3u_{00}v_{00}u_{01x} - 3v_{00}v_{01}u_{01x} + 3v_{10}u_{01x} + 3u_{00}u_{10x} +\\
& \quad 3v_{00}u_{11x} - 3u_{20x} + 3u_{01x}v_{00x} + 3u_{00}u_{00xx} +\\
& \quad 3v_{00}u_{01xx} - 3u_{10xx} - u_{00xxx} = 0,\\
& \quad \ldots\ldots\ldots\ldots
\end{aligned}$$

Let us show that the above lists of nonlinear PDE can be reduced to the integrated KP. First of all note that the reduction $\partial_2 \equiv \partial_1$ leads to the following identities:

$$v_{ij} \equiv u_{ij}, \quad u_{i1} \equiv u_{i0}, \quad \text{for all } \ i \ \text{ and } \ j. \qquad (71)$$

Then equations (66) and (67) becomes identities, while the system (68) - (70) results in three equations ($u = u_{00}$, $v = u_{10}$, $w = u_{20}$)

$$u_{t_1} + 4uu_x - 2v_x - u_{xx} = 0, \quad v_{t_1} + 4vu_x - 2w_x - v_{xx} = 0, \quad (72)$$

$$u_{t_3} - u_{xxx} - 3v_{xx} - 3w_x + 6u_x^2 + 6uu_{xx} + 6vu_x + 6uv_x - 12u^2 u_x = 0. \quad (73)$$

Eq.(73) results in the integrated KP after elimination of v and w:

$$u_{t_3} - \frac{1}{4}u_{xxx} - \frac{3}{4}\partial^{-1}u_{t_1 t_1} + 3u_x^2 = 0 \quad (74)$$

For this reason we call the above system (66) - (70) (together with the whole list of nonlinear PDE related with different variables t_γ) the generalization of KP hierarchy.

4 Construction of particular solutions

For construction of particular solutions to the nonlinear PDE which appear in this paper, one needs to choose the functions ψ_i, $i = 1, \ldots, Q_{n-1}$ which satisfy the conditions (5) and (6). One of the following Fourier integrals is suitable for this purpose.

For the equations from the Sec.2:

$$\psi_i = \int c_i(\mathbf{k}) \exp\left(\sum_{i=1}^{N} k_i x_i + \sum_\gamma \omega_\gamma(\mathbf{k}) t_\gamma\right) d\mathbf{k}, \quad (75)$$

$$\mathbf{k} = (k_1, \ldots, k_N), \quad i = 1, \ldots, Q_{n-1}.$$

Here ω_γ depends on $\mathbf{k}$ due to the dispersion equations associated with the eqs.(6)

$$\omega_\gamma = k^\gamma. \quad (76)$$

In this case ψ_i are arbitrary functions of all variables x_i.

For the equations from the Sec.3:

$$\psi_i = \int c_i(k_1) \exp\left(k_1 x_1 + \sum_{i=2}^{N} k_i(k_1) x_i + \sum_\gamma \omega_\gamma(k_1) t_\gamma\right) dk_1, \quad (77)$$

$$i = 1, \ldots, Q_{n-1}.$$

Functions $\omega_\gamma(k_1)$ and $k_i(k_1)$, $i>1$ represent the dispersion relations associated with eqs.(6) and (20):

$$\omega_\gamma = k^\gamma, \;\; k_i^{n_i+1} = \prod_{j=1}^{N} k_j^{\alpha_j^{(i)}}, \;\; a_i^{(i)} = 0, \;\; i = 2, 3, \ldots, N. \tag{78}$$

Here ψ_i are arbitrary functions of single variable x_1; $c_i(\mathbf{k})$ and $c_i(k_1)$ in formulas (75) and (77) are such functions of argument(s) that condition (5) is satisfied.

After this, one needs to solve the system of algebraic equations (8) for coefficients w^α_β which are the solutions of appropriate system of nonlinear PDE (16).

For instance, let us construct the particular solution for the system (48). In this case the equation (77) should be written in the form

$$\psi_i = \int c_i(k_2) \exp\left(k_2^2 x_1 + k_2 x_2 + k_2^4 t_1\right) dk_2, \;\; i = 1, \ldots, Q_{n-1}. \tag{79}$$

To construct the simple solution, let $Q_{n-1} = 3$ (or $n = 2$ due to the eq.(24)) and take the following expressions for ψ_i:

$$\psi_i = c_{1i} + c_{2i} \exp(k_i^2 x_1 + k_i x_2 + k_i^4 t_1), \;\; i = 1, 2, 3. \tag{80}$$

Then the systems (25) and (26) becomes 3×3 algebraic matrix equations for the coefficients u_{ij} and v_{ij} respectively, which can be solved directly. One can see that the condition (27) is satisfied for our choice of functions ψ_i. We give only expression for the function $u = v_{00}$:

$$\begin{aligned}
u &= \frac{s_1 E_1 E_2 + s_2 E_1 E_3 + s_3 E_2 E_3 + s_4 E_1 E_2 E_3}{p_1 E_1 E_2 + p_2 E_1 E_3 + p_3 E_2 E_3 + p_4 E_1 E_2 E_3}, \qquad (81)\\
E_i &= \exp(k_i^2 x + k_i y + k_i^3 t_1),\\
s_1 &= c_{13} c_{21} c_{22} k_1 k_2 (k_2^2 - k_1^2), \;\; s_2 = c_{12} c_{21} c_{23} k_1 k_3 (k_1^2 - k_3^2),\\
s_3 &= c_{11} c_{22} c_{23} k_2 k_3 (k_3^2 - k_2^2),\\
s_4 &= c_{21} c_{22} c_{23} k_1 k_2 (k_2 - k_1)(k_1 - k_3)(k_2 - k_3)(k_1 + k_2 + k_3),\\
p_1 &= c_{13} c_{21} c_{22} k_1 k_2 (k_1 - k_2), \;\; p_2 = c_{12} c_{21} c_{23} k_1 k_3 (k_3 - k_1),\\
p_3 &= c_{11} c_{22} c_{23} k_2 k_3 (k_2 - k_3), \;\; p_4 = c_{21} c_{22} c_{23} (k_1 - k_2)(k_1 - k_3)(k_2 - k_3),
\end{aligned}$$

If all coefficients p_i in the denominator of this solution have the same sign (this can be arranged by the appropriate choice of parameters c_i and k_i), then the solution has no singularities for all values of independent variables and describes inelastic three-kink interaction.

5 Conclusions

We represent an example of multidimensional generalization of Sato equation (17), written in terms of pure differential operators. We give algorithm for construction of different multidimensional hierarchies which admit the special family of particular solutions. Namely, solutions are parameterized by the set of arbitrary functions of either one or several variables x_i. Different subspaces of general space (3) result in specific systems of nonlinear PDE. Two examples of this kind are given in the Sec.3.

The represented approach might be considered as the mixture of Sato theory and Lax representation. In fact, each equation of the system (17) is nothing but the compatibility condition of the overdetermined linear systems (6) and (7). For each particular γ, the equation (17) gives the complete system of nonlinear PDE (16) on the coefficients w^α_β of the operators W_α. But the structure of this system depends on n (order of differential operators S_α) in general case. The examples of systems of nonlinear PDE which do not depend on n are given in the Sec.3.

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